Vector Metric Spaces and Its Completion

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Abstract: Our aim in this study is to show distance that is known as scalar size, can be thought as vectorial size. We introduce vector metric spaces and determine their completions.

Keywords: Vector metric space, order-Cauchy sequence, order-Cauchy completion, Riesz space, Banach lattice.

1. INTRODUCTION

Let $E$ be a Riesz space. If $(a_n)$ is a decreasing sequence in $E$ such that \(\inf a_n = a\), we will write $a_n \downarrow a$. A sequence $(b_n)$ is said to order convergence (or $\omega$-convergence) to $b$ if there is a sequence $(a_n)$ in $E$ satisfying $a_n \downarrow 0$ and $|b_n - b| \leq a_n$ for all $n$, and written $b_n \to^\omega b$ or $\lim b_n = b$. Furthermore $(b_n)$ is said to be order-Cauchy (or $\omega$-Cauchy) if there exists a sequence $(a_n)$ in $E$ such that $a_n \downarrow 0$ and $|b_n - b_{n+p}| \leq a_n$ holds for all $n$ and $p$. $E$ is said to be $\omega$-Cauchy complete if every $\omega$-Cauchy sequence is $\omega$-convergence. For notations and other facts regarding Riesz spaces we refer to (1) and (7).

We replace the real numbers by Riesz space and define vector metric spaces $(X, d, E)$. In Section 2, we define basic concepts of the metric spaces theory in vector metric spaces. Then we obtain some new results in the vector metric spaces.

Section 3 is an exposition of the fundamentals of completeness and completion by $E$-Cauchy sequences in vector metric spaces. In (2), order-Cauchy completeness and completions of Archimedean Riesz spaces and rings of real-valued continuous functions are discussed. Most of the material presented in (2) is due to Everett (3) and Papangelou (6) in the case of $l$-groups. Every Archimedean Riesz space or $f$-algebra has an $\omega$-Cauchy completion in a sense defined precisely in (2). Among other things, conditions are given under which the order-Cauchy completion and the Dedekind completion coincide. Completions akin to the order-Cauchy completion are described also in (8). We give an abstract characterization of $E$-completion for vector metric which valued $\omega$-Cauchy complete Riesz space.

2. VECTOR METRIC SPACES

In this section we define vector metric spaces and prove some properties.

Definition 1. Let $X$ be a nonempty set and $E$ be a Riesz space. The function $d : X \times X \to E$ is said to be a vector metric (or $E$-metric) if it is satisfying the following properties:

\[(\text{vm1})\] $d(x, y) = 0$ if and only if $x = y$.

\[(\text{vm2})\] $d(x, y) \leq d(x, z) + d(y, z)$

for all $x, y, z \in X$. Also the triple $(X, d, E)$ is said to be vector metric space.

Proposition 1. For arbitrary elements $x, y, z, w$ of a vector metric space, the following properties hold:

(i) $0 \leq d(x, y)$;

(ii) $d(x, y) = d(y, x)$;

(iii) $|d(x, z) - d(y, z)| \leq d(x, y)$;

(iv) $|d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)$.

Proof. (i) and (ii) are easy.

(iv) From

$d(x, z) - d(y, w) \leq d(x, y) + d(z, w)$

it follows that

$|d(x, z) - d(y, w)| = |d(x, z) - d(y, w)] \vee |d(y, w) - d(x, z)|$

$\leq d(x, y) + d(z, w)$.

(iii) Using $(\text{vm1})$, we take $z = w$ in (iv), we have (iii).

Now we give some examples of vector metric spaces.

Example. (a) A Riesz space $E$ is a vector metric space with $d : E \times E \to E$ defined by $d(x, y) = |x - y|$. This vector metric is called to be absolute valued metric on $E$.

(b) The complexification $E_C$ of a real Banach lattice $E$ is a vector metric space with $d : E_C \times E_C \to E$ defined by $d(x_C, y_C) = |x_C - y_C|$.

(c) Suppose we have a finite number of vector metric spaces $(X_i, d_i, E)$, $i = 1, \ldots, k$. On the cartesian product $X = X_1 \times \cdots \times X_k$ various vector metrics can be defined. Let $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ be two elements of the product space $X$. We define

\[
d_{11}(x, y) = \sum_{i=1}^{k} d_i(x_i, y_i);
\]

\[
d_{\infty}(x, y) = \sup \{d_i(x_i, y_i) : i = 1, \ldots, k\}.
\]
It can be verified easily that \( d_{(1)} \) and \( d_\infty \) are vectorial distance functions on the product space \( X \). If \( E \) is Banach lattice and \( 1 < p < \infty \) where \( \frac{1}{p} + \frac{1}{q} = 1 \), then by the functional calculus of J.L. Krivine, \( \sum_{i=1}^{k} d_i(x_i, y_i)^p \in E \) with
\[
\left( \sum_{i=1}^{k} (d_i(x_i, y_i))^p \right)^{\frac{1}{p}} = \sup \left\{ \sum_{i=1}^{k} |a_i| d_i(x_i, y_i) : (a_1, \ldots, a_k) \in \mathbb{R}^k, \sum_{i=1}^{k} |a_i|^q \leq 1 \right\}
\]
(see (4), (5, pp. 42-44)). We define
\[
d_{(p)}(x, y) = \left( \sum_{i=1}^{k} (d_i(x_i, y_i))^p \right)^{\frac{1}{p}}.
\]
Thus, \( d_{(p)} \) is a vector metric on the product space \( X \).

**Definition 2.** (a) A sequence \( (x_n) \) in a vector metric space \( (X, d, E) \) vectorial converges (or is \( E \)-converges) to some \( x \in E \), written \( x_n \to d, E \) \( x \), if there is a sequence \( (a_n) \) in \( E \) satisfying \( a_n \downarrow 0 \) and \( d(x_n, x) \leq a_n \) for all \( n \).

(b) A sequence \( (x_n) \) is called an \( E \)-Cauchy sequence whenever there exists a sequence \( (a_n) \) in \( E \) such that \( a_n \downarrow 0 \) and \( d(x_n, x_{n+p}) \leq a_n \) holds for all \( n \) and \( p \).

(c) A vector metric space \( (X, d, E) \) is called \( E \)-complete if each \( E \)-Cauchy sequence in \( X \) \( E \)-converges to a limit in \( X \).

(d) A subset \( Y \) of a vector metric space \( (X, d, E) \) is said to be \( E \)-closed whenever \( (x_n) \subseteq Y \) and \( x_n \to d, E \) \( x \) imply \( x \in Y \).

**Remark.** More explicit (and overly cumbersome) terminology would perhaps be sequentially \( E \)-Cauchy complete, to distinguish from the corresponding notion for nets. However, this paper is concerned exclusively with sequences, so dropping “sequentially” introduces no ambiguity here.

**Theorem 2.** If \( x_n \to d, E \) \( x \), then the following hold:
(i) The limit \( x \) is unique.
(ii) Every subsequence of \( (x_n) \) \( E \)-converges to \( x \).
(iii) If also \( y_n \to d, E \) \( y \), then \( d(x_n, y_n) \to d(x, y) \).

Also, we have the following theorem.

**Theorem 3.** For the vector metric space \( (X, d, E) \), the followings hold:
(i) Every \( E \)-convergent sequence is an \( E \)-Cauchy sequence;
(ii) Every \( E \)-Cauchy sequence is \( E \)-bounded;
(iii) If an \( E \)-Cauchy sequence \( (x_n) \) has a subsequence \( (x_{n_k}) \) such that \( x_{n_k} \to d, E \) \( x \), then \( x_n \to d, E \) \( x \);
(iv) If \( (x_n) \) and \( (y_n) \) are \( E \)-Cauchy sequences, then \( (d(x_n, y_n)) \) is an order Cauchy sequence.

**Proof.** (i) Let \( x_n \to d, E \) \( x \) in \( X \). Since there exists a sequence \( (a_n) \) in \( E \) such that \( a_n \downarrow 0 \) and \( d(x_n, x_{n+p}) \leq d(x_n, x) + d(x_{n+p}, x) \leq a_n + a_{n+p} \leq 2a_n \) for all \( n \) and \( p \), then \( (x_n) \) is an \( E \)-Cauchy sequence in \( X \).

(ii) Let \( (x_n) \) be an \( E \)-Cauchy sequence in \( X \). Since there exists a sequence \( (a_n) \) in \( E \) such that \( d(x_n, x_{n+p}) \leq a_n \downarrow 0 \) for all \( n \) and \( p \), then \( d(x_n, x_{n+p}) \leq a_1 \), that is, \( (x_n) \) is \( E \)-bounded in \( X \).

(iii) Let \( (x_n) \) be an \( E \)-Cauchy sequence and let \( (x_{n_k}) \) be a subsequence of \( (x_n) \) such that \( x_{n_k} \to d, E \) \( x \) in \( X \). If we take \( n_k = n + p \), where \( n \leq n_k \) for all \( n \), then there exist two sequences \( (a_n) \) and \( (b_n) \) in \( E \) such that
\[
d(x_n, x) \leq d(x_n, x_{n+p}) + d(x, x_{n+p})
\]
and hence \( x_n \to d, E \) \( x \).

(iv) Since there exist two sequence \( (a_n) \) and \( (b_n) \) in \( E \) such that
\[
|d(x_n, y_n) - d(x_{n+p}, y_{n+p})| \leq d(x_n, x_{n+p}) + d(y_n, y_{n+p}) \leq a_n + b_n \downarrow 0
\]
for all \( n \) and \( p \), then the sequence \( (d(x_n, y_n)) \) is an \( E \)-Cauchy sequence in \( E \).

When \( E = R \), the concepts of vectorial convergence and convergence in metric are the same. When also \( X = E \) and \( d \) is the concepts of absolute valued vector metric, vectorial convergence and convergence in order are the same. When \( E = R \), the concepts of \( E \)-Cauchy sequence and Cauchy sequence are the same.

Now, let us fix a vector metric space \( (X, d, E) \). For two elements \( a \) and \( b \) in \( E \), we shall write \( a < b \) to indicate that \( a \leq b \) but \( a \neq b \), while \( b > a \) will stand for \( a < b \).

**Definition 3.** (a) A subset \( Y \) of \( X \) is called \( \tau_{d, E} \)-dense whenever \( B(x, r) \cap Y \neq \emptyset \) for each \( x \in X \) and \( 0 < r \in E \).

(b) A subset \( Y \) of \( X \) is called \( E \)-dense whenever for every \( x \in X \) there exists a sequence \( (x_n) \) in \( Y \) satisfying \( x_n \to d, E \) \( x \).

We have already following results.

**Corollary 4.** Let \( Y \) be a subset of a vector metric space \( (X, d, E) \) with \( E \) Archimedean. If \( Y \) is \( \tau_{d, E} \)-dense, then \( Y \) is \( E \)-dense.

The relationships between the concepts of boundedness and diameter of a subset of a vector metric space are different from the usual. For a nonempty subset \( A \) of a vector metric space \( (X, d, E) \) its \( E \)-diameter defined by \( d(A) = \sup \{ d(x, y) : x, y \in A \} \) if \( \sup \{ d(x, y) : x, y \in A \} \) in \( E \). Furthermore, if there exists an \( a > 0 \) in \( E \) such that \( d(x, y) \leq a \) for \( x, y \in A \), then \( A \) is called \( E \)-bounded set. If \( E \) is Dedekind complete, then every \( E \)-bounded set of \( (X, d, E) \) has a diameter.
3. ORDER-CAUCHY COMPLETIONS OF VECTOR METRIC SPACES

A map \( i : X \to Y \) between two vector metric spaces \((X, d, E)\) and \((Y, \rho, E)\) is called \( E \)-isometry if \(d(x, y) = \rho(i(x), i(y))\) holds for all \(x, y \in X\). If in addition \(i\) is onto, then \((X, d, E)\) and \((Y, \rho, E)\) are called \(E\)-isometric. Given a vector metric space \((X, \tilde{d}, E)\) which is not \(E\)-complete, we are going to construct a \(E\)-complete vector metric space \((\tilde{X}, \tilde{d}, E)\) such that there is an \(E\)-isometry \(i : X \to \tilde{X}\) with the property that \(i(X)\) is \(E\)-dense in \(\tilde{X}\). We call \((\tilde{X}, \tilde{d}, E)\) the \(E\)-Cauchy completion of \((X, d, E)\).

Let \(E\) be an \(E\)-Cauchy complete Riesz space and let \((x_n)\) and \((y_n)\) be two \(E\)-Cauchy sequences in \(X\). Then we define a equivalence relation denoted by

\[
d(x_n, y_n) \to 0.
\]

Let \(\tilde{X}\) be the set of all equivalence classes of \(E\)-Cauchy sequences of \(X\). Since \((\tilde{d}(x_n, y_n))\) is an order Cauchy sequence whenever \((x_n)\) and \((y_n)\) are \(E\)-Cauchy sequences, by \(E\) is order-Cauchy we define

\[
d(\tilde{x}, \tilde{y}) = o - \lim d(x_n, y_n).
\]

for all \(\tilde{x}, \tilde{y} \in \tilde{X}\) with \((x_n) \in \tilde{x}\) and \((y_n) \in \tilde{y}\). It is not difficult to see that \((\tilde{X}, \tilde{d}, E)\) is a vector metric space.

For an element \(x\) of \(X\) consider the equivalence class \(i(x) \in \tilde{X}\) which is generated by the constant sequence whose each term is \(x\). This way we define a map \(i : X \to \tilde{X}\).

Then \((\tilde{X}, \tilde{d}, E)\) and \((i(X), \tilde{d}, E)\) are \(E\)-isometric.

**Lemma 5.** \((i(X), \tilde{d}, E)\) is \(E\)-dense in \((\tilde{X}, \tilde{d}, E)\).

**Proof.** Let \((x_n) \in \tilde{x}\) and for fixed \(n_0\) consider \(i(x_{n_0})\) as an element \(i(X)\). \(i(x_{n_0})\) is of course the equivalence class generated by the constant \((x_{n_0}, x_{n_0}, \ldots)\). For every \(n \geq n_0\) there exists a number \(p\) such that \(n = n_0 + p\). Since \((x_n)\) is \(E\)-Cauchy, there exists a sequence \((a_n)\) in \(E\) such that \(a_n \to 0\) and

\[
d(i(x_{n_0}), \tilde{x}) = o - \lim d(x_{n_0}, x_n) = o - \lim d(x_{n_0}, x_{n_0+p}) \leq a_n
\]

holds for all \(n \geq n_0\) with \(n = n_0 + p\). Hence \(d(i(x_k), \tilde{x}) \leq a_k\) for all \(k\); i.e., \(i(x_k) \to \tilde{x}\).

We have following theorem.

**Theorem 6.** \((\tilde{X}, \tilde{d}, E)\) is unique \(E\)-complete vector metric space generated by \((X, d, E)\).

**Proof.** Let \((\tilde{x}_n)\) be a \(E\)-Cauchy sequence in \(\tilde{X}\). Using the fact that \(i(X)\) is \(E\)-dense in \(\tilde{X}\) for each \(n\) we find \(y_n\) in \(X\) and \(a_n\) in \(E\) with

\[
d(\tilde{i}(y_n), \tilde{x}_n) \leq a_n.
\]

We will show that \((y_n)\) is a \(E\)-Cauchy sequence in \(X\) and if \(\tilde{y}\) is the equivalence class generated by \((y_n)\), then \(\tilde{x}_n \to \tilde{y}\) in \((\tilde{X}, \tilde{d}, E)\).

We have

\[
d(y_n, y_{n+p}) = d(i(y_n), i(y_{n+p})) \\
\leq d(i(y_n), \tilde{x}_n) + d(\tilde{x}_n, x_{n+p}) + d(x_{n+p}, i(y_{n+p})) \\
\leq 3a_n
\]

for all \(n\) and \(p\). By (1) we have \(i(y_n) \to \tilde{d}, E, \tilde{y}\). Hence,

\[
d(\tilde{y}, \tilde{x}_n) \leq d(\tilde{y}, i(y_n)) + d(i(y_n), \tilde{x}_n) \leq 2a_n
\]

which implies \(\tilde{x}_n \to \tilde{d}, E, \tilde{y}\). Therefore, \((\tilde{X}, \tilde{d}, E)\) is \(E\)-complete.

Let \((X, \tilde{d}, E)\) be another \(E\)-complete vector metric space such that there is an \(E\)-isometry \(j : X \to \tilde{X}\) with \(j(x)\) is \(E\)-dense in \(\tilde{X}\). We will show that \((X, \tilde{d}, E)\) is then isometrically equivalent to the completion \((\tilde{X}, \tilde{d}, E)\). Hence in this sense the completion is unique.

We define first \(h_0 : i(X) \to j(X)\) by \(h_0(i(x)) = j(x)\) for all \(x \in X\). It is clear that \(i(X)\) and \(j(X)\) are \(E\)-isometric. Let \(\tilde{x} \in \tilde{X}\). Then there exists a \(E\)-Cauchy sequence \(i(x_n)\) in \(i(X)\) such that \(i(x_n) \to \tilde{d}, E, \tilde{x}\). Since \(h_0\) is \(E\)-isometry, then \(j(x(n)) = (h_0 \circ i)(x_n)\) is also \(E\)-Cauchy sequence in \(\tilde{X}\). Because \(\tilde{X}\) is \(E\)-complete, there exists \(\tilde{x}\) in \(\tilde{X}\) such that \(j(x(n)) \to \tilde{d}, E, \tilde{x}\). Hence we define

\[
h := \tilde{X} \to \tilde{X} \text{ with } h(\tilde{x}) = \tilde{x}.
\]

For two elements \(\tilde{x}\) and \(\tilde{y}\) in \(\tilde{X}\) there exist two sequences \((i(x_n))\) and \((i(y_n))\) in \(i(X)\) such that \(i(x_n) \to \tilde{d}, E, \tilde{x}\) and \(i(y_n) \to \tilde{d}, E, \tilde{y}\). Then \(j(x(n)) \to \tilde{d}, E, h(\tilde{x})\) and \(j(y(n)) \to \tilde{d}, E, h(\tilde{y})\) hold. Since \(h_0\) is \(E\)-isometry, by Theorem 2 (4) we have

\[
d(h(\tilde{x}), h(\tilde{y})) = o - \lim d(j(x_n), j(y_n)) \\
= o - \lim d(i(x_n), i(y_n)) \\
= \tilde{d}(\tilde{x}, \tilde{y}).
\]

Finally, we will show that \(h\) is onto. If \(\tilde{x}\) in \(\tilde{X}\), then there exists a \(E\)-Cauchy sequence \(j(x_n)\) in \(j(X)\) such that \(j(x_n) \to \tilde{x}\). Since \(i(X)\) \(E\)-isometric to \(j(X)\), \(i(x_n)\) is also a \(E\)-Cauchy sequence in \(\tilde{X}\). If \(\tilde{x}\) is \(\tilde{d}\)-limit of \(i(x_n)\), then \(h(i(x_n)) \to \tilde{d}, E, h(\tilde{x})\) since \(h\) is \(E\)-isometry. By using \(j(x_n) = h(i(x_n))\) we have

\[
d(\tilde{x}, h(\tilde{x})) \leq d(\tilde{x}, j(x(n)) + d(h(\tilde{x}), h(i(x(n))))
\]

for each \(n\). Therefore, \(h(\tilde{x}) = \tilde{x}\) holds in \(\tilde{X}\).

**REFERENCES**


