# Subdiffusion in a carious lesion process ⋆

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**Abstract:** We study a progress of carious lesion caused by organic acids diffusing into the tooth enamel from dental plaque. The acids react with static hydroxyapatite, which leads to demineralization of the enamel, and consequently to development of caries. Theoretical model of this process utylizes nonlinear subdiffusion–reaction equations with fractional time derivative. We find the time evolution of the reaction front, that can be identified with the depth of caries, and we determine the subdiffusion parameter of acids in the tooth enamel.

Keywords: Biomedical systems, nonlinear models, interdisciplinary design, complex systems, parameter estimation.

## 1. INTRODUCTION

Subdiffusion is defined as a process where the mean square displacement of a particle $\langle \Delta x^2 \rangle$ is a power function of time

$$\langle \Delta x^2(t) \rangle = \frac{2D_\alpha}{\Gamma(1+\alpha)} t^\alpha, \tag{1}$$

$\alpha$ is the subdiffusion parameter Metzler and Klafter (2000). The subdiffusion occurs in systems with complex internal structure such as gels or porous media. One of the systems where subdiffusion can occur is the tooth enamel. The caries is caused by organic acids which diffuse into the interior of the enamel from dental plague and chemically react with static hydroxyapatite, which is the main component of the enamel. Theoretical model of this process is based on the nonlinear subdiffusion-reaction equations containing a fractional time derivative with one static substrate; we additionally assume that the reagents are separated form each other at the initial moment.

## 2. CARIES LESION PROGRESS

Firstly, we present the model of caries based on the paper by Lewandowska and Kosztołowicz (2007) and references cited therein. The enamel consists of hydroxyapatite crystals
$Ca_{10}(PO_4)(OH)_2$ in $92-94\%$ by weight. These crystals are organized in larger forms called prisms. The intercrystalline and interprismatic spaces of enamel are filled with water. Because of spaces between crystals and prisms, enamel is a microporous material. Besides *HA* the enamel includes inorganic factors, mostly fluoride and carbonate. In addition more then 40 trace elements can occur in the tooth mineral. Organic matrix represents less then 1% by weight of the enamel. The surface of dental enamel is covered by the dental plaque which mainly consists of microorganisms, saliva, leftovers and mucus. Oral microorganisms metabolize simple sugars coming from diet to the organic acids (e.g., acetic or lactic). The formation of carious lesion of the enamel starts when the concentration of organic acids in the dental plaque reaches sufficient value and *pH* of the dental plaque lowers below the appropriate point. Then the organic acids diffuse into the enamel. The acids can be transported in dissociated or undissociated form that depends on *pH* of the dental plaque. After achieving the enamel interior, acid reacts with the mineral according to the chemical formula

$$Ca_{10}(PO_4)_6(OH)_2 + 8H^+ \rightarrow 10Ca^{2+} + 6HPO_4^{2-} + 2H_20,$$

where the phosphate ions have an acidic form determined by the *pH* of the system. The products of reaction are inert for the caries progress. It is commonly accepted that the products of the reaction, calcium ions and phosphate ions (or complexes), are transported out of the enamel by the diffusion. Host factors involved in the caries process are as follows: composition and structure of the enamel, *pH* and buffer concentration, kind of acid, mineral content gradient, composition of saliva, age of tooth, environment, diet, hygiene, etc.

## 3. SUBDIFFUSION–REACTION EQUATIONS

A real system is usually three–dimensional, but we assume that it is homogeneous in the plane perpendicular to the $x$ axis. Therefore, we take one space variable $x$ into consideration. The subdiffusion–reaction equations are given by following equations with Caputo fractional derivative

$$\frac{\partial^\alpha}{\partial t^\alpha} C_A(x,t) = D_A \frac{\partial^2}{\partial x^2} C_A(x,t) - R(x,t), \tag{2}$$

$$\frac{\partial^\alpha}{\partial t^\alpha} C_B(x,t) = D_B \frac{\partial^2}{\partial x^2} C_B(x,t) - R(x,t), \tag{3}$$

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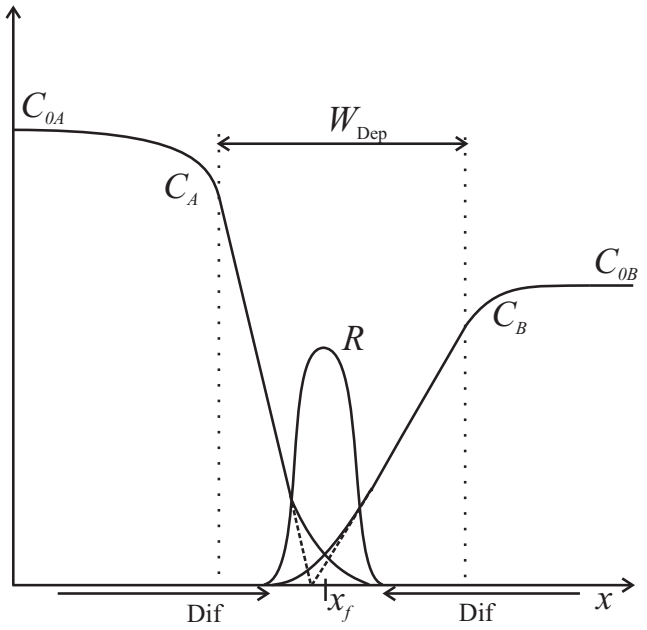


Fig. 1. Schematic view of the system under considerations, $x_f$ is the reaction front, $W_{\mathrm{Dep}}$ denotes the depletion zone.

where $C_i$ denotes the concentration of the diffusing particles of species $i$, $D_i$ – the subdiffusion coefficient, $i = A, B$. Within the mean field approximation for the case of chemical reaction $m'A + n'B \to P$ the term $R$ reads

$$R(x,t) = kC_A^m(x,t)C_B^n(x,t), \tag{4}$$

where $k$ is the reaction rate and the parameters $m$ and $n$ are determined experimentally. We assume that the particles of reactants $A$ and $B$ are initially separated from each other. Thus, the initial conditions are

$$C_A(x,0) = \begin{cases} C_{0A}, & x > 0 \\ 0, & x < 0 \end{cases}, \qquad C_B(x,0) = \begin{cases} 0, & x > 0 \\ C_{0B}, & x < 0 \end{cases}. \tag{5}$$

It is observed Gàlfi and Ràcz (1988); ben Avraham and Havlin (2000) that when the process starts, few characteristic regions appear (see Fig. 1): the depletion zone Dep, which is defined as a region where the concentrations are significantly smaller than the initial ones ($C_A \ll C_{0A}$ and $C_B \ll C_{0B}$), the reaction region R where the production of particles $P$ is significant ($R(x,t) > 0$), and the diffusion region Dif, where the reaction term $R(x,t)$ is close to zero and the particle transport appears to be almost diffusive (i.e. without chemical reactions).

The solutions of the equations have not been found yet. To describe the process some characteristic functions are used. One of them is the time evolution of the reaction front $x_f$, which is defined as a point where the reaction term reaches its maximum.

## 4. TIME EVOLUTION OF REACTION FRONT

Using the quasistatic approximation we showed that in a subdiffusion–reaction system with arbitrary non–zero values of subdiffusion coefficients the reaction front evolves in time according to the formula Kosztołowicz and Lewandowska (2006)

$$\Phi\left(\frac{-x_f(t)}{\sqrt{D_A t^\alpha}}\right) = \frac{n}{m}\frac{\sqrt{D_A}C_{0A}}{\sqrt{D_B}C_{0B}}\Phi\left(\frac{x_f(t)}{\sqrt{D_B t^\alpha}}\right), \tag{6}$$

where $\Phi(z) \equiv H_{11}^{10}\left(z\middle|\begin{matrix}1 & \alpha/2\\ 0 & 1\end{matrix}\right)/Q(z)$,

$$Q(z) = \frac{\alpha}{2}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\alpha/2(1-k))}(-z)^k\,,$$

$H$ denotes the Fox function which can be expressed as

$$H_{11}^{10}\left(z\middle|\begin{matrix}1 & \alpha/2\\ 0 & 1\end{matrix}\right) = \sum_{k=0}^{\infty}\frac{1}{k!\Gamma(1-\alpha k/2)}(-z)^k.$$

The solution of (6) is

$$x_f(t) = Kt^{\alpha/2}, \tag{7}$$

where coefficient $K$ fulfills the following equation

$$\Phi\left(\frac{-K}{\sqrt{D_A}}\right) = \frac{n}{m}\frac{\sqrt{D_A}C_{0A}}{\sqrt{D_B}C_{0B}}\Phi\left(\frac{K}{\sqrt{D_B}}\right), \tag{8}$$

To find the time evolution of the reaction front for the system with static substrate $B$, we take the limit of $D_{\alpha B} \to 0$. Expressing $Q$ by $H$ Kosztołowicz and Lewandowska (2006) and using the following approximation of the Fox functions for large argument Srivastava et al. (1982)

$$H_{11}^{10}\left(u\middle|\begin{matrix}a_1 & a_2\\ b_1 & b_2\end{matrix}\right) \sim \exp(-\delta|u|^{1/\delta}\gamma^{1/\delta})|u|^{(\epsilon+1/2)/\delta}, \tag{9}$$

where $\delta = b_2 - a_2$, $\gamma = a_2^{a_2}b_2^{-b_2}$ and $\epsilon = b_1 - a_1$, we get

$$\Phi\left(\frac{-x_f(t)}{\sqrt{D_A t^\alpha}}\right) = \frac{n}{m}\frac{\alpha\sqrt{D_A t^\alpha}C_{0A}}{2\sqrt{C_{0B}}x_f(t)}. \tag{10}$$

The solution of (10) is (7) where $K$ fulfills the relation

$$\Phi\left(\frac{-K}{\sqrt{D_A}}\right) = \frac{n}{m}\frac{\alpha\sqrt{D_A}C_{0A}}{2\sqrt{C_{0B}}K}. \tag{11}$$

Thus, the time evolution of the reaction front is the power function with the exponent depending on the subdiffusion parameter $\alpha$ only; the subdiffusion coefficient $D_A$ controls the parameter $K$.

The procedure developed in this paper is an extension of the one already used for the normal diffusion case Koza (1997). Repeating our consideration for $\alpha = 1$ we obtain the results identical with those from Koza (1997).

## 5. SUBDIFFUSION IN THE TOOTH ENAMEL

As an application of our theoretical results we consider the time evolution of the depth of carious lesion in human tooth enamel. The formation of caries is a result of the reaction of organic acids (lactic or acetic) with hydroxyapatite. Acids are transported from the outer environment to the interior of the static enamel. The reaction takes place in a restricted region of the enamel called the reaction zone. At the beginning the zone is placed

right below the outer enamel surface but after exhausting the hydroxyapatite in that region the reaction zone moves inside the enamel. The depth of lesion can be defined as a distance from the outer enamel surface to the boundary of the region penetrated by organic acids or as the distance from the surface to the boundary of the region where the hardness value is lower in comparison to the sound enamel. But, regardless of a choice of the definition of the lesion depth, it always occurs within the reaction zone. So, the reaction front $x_f$ can be identified with the depth of carious lesion. Its time evolution is measured experimentally. The experiments are performed *in vitro* with extracted teeth being demineralized in buffer solutions of organic acids at different values of pH. The time evolution of the depth of caries $\delta$ is found as the power function of time $\delta(t) = At^{\epsilon}$. We determine the exponent $\epsilon = \alpha/2$ of this function on the basis of the experimental data presented in Featherstone et al. (1978, 1979); Holly and Gary (1968).

The interval of extracted values of $\epsilon$, which is $(0,3; 0.5)$, is rather broad, but physical reasons of it were not explained in the papers cited above. The width of the interval $(0.3; 0.5)$ is probably caused by the kind of a tooth, its age and individual immunity, the applied prevention, the kind of buffer solution, its concentration and pH. The values close to 0.3 are mainly observed for the teeth in which the carious lesion is slight. This result is not surprising since the relatively large molecules of organic acids are usually transported in undissociated form inside the enamel which is of porous structure. When the caries is in progress, the size of enamel pores enlarges and the subdiffusion parameter increases. When the caries was caused by lactic acid molecules $\epsilon = 0.38$ Featherstone et al. (1979), which corresponded to $\alpha = 0.76$. The subdiffusion case $\epsilon < 0.5$ was observed also for acetic acid Featherstone and Rodgers (1981) (unfortunately, the errors of experimental data were not given in the papers cited above). Thus, for organic acids the subdiffusion is observed inside the tooth enamel.

## REFERENCES


D. ben Avraham and S. Havlin. *Diffusion and Reactions in Fractals and Disordered Systems*. Cambridge University Press, Cambrige, first edition, 2000.

J.D. Featherstone and B.E. Rodgers. The effect of acetic, lactic and other organic acids on the formation of artificial carious lesions. *Caries Res.*, 15:377–385, 1981.

J.D.B. Featherstone, J.F. Duncan, and T.W. Cutress. Surface layer phenomena in in–vitro early caries–like lesions of human tooth enamel. *Archs. Oral Biol.*, 23: 397–404, 1978.

J.D.B. Featherstone, J.F. Duncan, and T.W. Cutress. A mechanism for dental caries based on chemical processes and diffusion phenomena during in-vitro caries simulation on human tooth enamel. *Archs. Oral Biol.*, 24: 101–112, 1979.

L. Gàlfi and Z. Ràcz. Properties of reaction front in an $a + b \to c$ type reaction-diffusion process. *Phys. Rev. A*, 38(6):3151–3154, 1988.

F.J. Holly and J.A. Gary. Mechanism for incipient carious lesion growth utilizing a physical model based on diffusion concepts. *Archs. Oral Biol.*, 13:319–334, 1968.

T. Kosztołowicz and K.D. Lewandowska. Time evolution of the reaction front in a subdiffusive system. *http://arxiv.org/archive/cond-mat*, cond-mat/0603139: 1–18, 2006.

Z. Koza. Asymptotic behaviour of initially separated $a + b(static) \to 0$ reaction-diffusion system. *Physica A*, 240: 622–634, 1997.

K.D. Lewandowska and T. Kosztołowicz. Perturbation method to model enamel caries progress. *arxiv*, 0707.0194:1–8, 2007.

R. Metzler and J. Klafter. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.*, 339(1):1–77, 2000.

H.M. Srivastava, K.C. Gupta, and S.P. Goyal. *The H-functions of one and two variables with applications*. South Asia Publ., New Delhi, 1982.